https://www.linkedin.com/feed/update/urn:li:activity:6460531442408660992

Solve for x

$$x = \sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{1 + \ldots}}}}$$

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Let $r_n := \sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{1 + \ldots + n\sqrt{1}}}}}\ , n \in \mathbb{N}$.

We will prove that $\lim\limits_{n\to\infty} r_n = 3.$

Let $r_n(p) = \sqrt{1 + p\sqrt{1 + (p+1)\sqrt{1 + (p+2)\sqrt{1 + \ldots + n\sqrt{1}}}}}\ , n,p \in \mathbb{N}$ and $p \leq n$, then

$r_n(p) < p + 1,$ Indeed, if $n = p$ then $r_p(p) = \sqrt{1 + p\sqrt{1}} = \sqrt{1 + p} < p + 1$ and from supposition

$r_n(p) < p + 1$ for any $n,p \in \mathbb{N}$ follows $r_{n+1}(p) = \sqrt{1 + pr_n(p+1)} < \sqrt{1 + p(p+2)} = p + 1.$

Thus, $r_n = r_n(2) < 3$ and

$$0 < 3 - r_n = \frac{9 - r_n^2(2)}{3 + r_n(2)} = \frac{2(4 - r_n(3))}{3 + r_n(2)} = \frac{2(15 - 3r_n(4))}{(3 + r_n(2))(4 + r_n(3))} =$$

$$\frac{2 \cdot 3(5 - r_n(4))}{(3 + r_n(2))(4 + r_n(3))} = \ldots = \frac{2 \cdot 3 \cdot \ldots \cdot ((n+1) - r_n(n))}{(3 + r_n(2))(4 + r_n(3))\ldots(n + 1 + r_n(n))} =$$

$$\frac{2 \cdot 3 \cdot \ldots \cdot (n^2 + 2n - n)}{(3 + r_n(2))(4 + r_n(3))\ldots(n + 1 + r_n(n))} = \frac{2 \cdot 3 \cdot \ldots \cdot n \cdot (n+1)}{(3 + r_n(2))(4 + r_n(3))\ldots(n + 1 + r_n(n))} <$$

$$\frac{2 \cdot 3 \cdot \ldots \cdot (n+1)}{(3+1)(4+1)\ldots(n+1+1)} = \frac{6(n+1)!}{(n+2)!} = \frac{6}{n+2}.$$ So, $\lim\limits_{n\to\infty} r_n = 3.$